CS/SE4-6TE3, CES 722/723: Tutorial 6

October 26, 2010

- **Conjugate Vectors**

Give three vectors a, b and c so that the three vectors form a conjugate system of vectors w.r.t. the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 9 \end{pmatrix}.$$

We can fix the first vector. Let $a = (1,0,0)^T$, then $a^T A = (1,1,1)$. By inspecting $a^T A$ we can set the second vector to $b = (1,1,-2)^T$, since $a^T Ab = 0$ iff a and b are conjugate to symmetric matrix A. Then we can set $c = (x,y,z)^T$ and find a solution for it.

$$\begin{cases} a^T Ac = 0 \\ b^T Ac = 0 \end{cases} \Leftrightarrow \begin{cases} x+y+z=0 \\ 2y-16z=0 \end{cases} \Leftrightarrow \begin{cases} x+y+z=0 \\ y=8z \end{cases} \Leftrightarrow \begin{cases} x=-9z \\ y=8z \end{cases}$$

We can assign any value to z. Let'ss say that we pick $z = 1$, then the three conjugate vectors we find are $a = (1,0,0)^T$, $b = (1,1,-2)^T$ and $c = (-9,8,1)^T$.

- **Example to apply the Conjugate Gradient Method**

Minimize $q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

starting from $\mathbf{x}_1 = (0,0)^T$.

Solution:

Initial search direction:

$$\mathbf{s}_1 = -\mathbf{g}_1 = -\nabla q(\mathbf{x}_1) = \mathbf{b} - \mathbf{A}\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step size:

$$\lambda_1 = \frac{-\mathbf{g}_1^T\mathbf{s}_1}{\mathbf{s}_1^T\mathbf{A}\mathbf{s}_1} = \frac{2}{7}$$

Next point:

$$\mathbf{x}_2 = \mathbf{x}_1 + \lambda_1\mathbf{s}_1 = \begin{pmatrix} 2/7 \\ 2/7 \end{pmatrix}$$

Next search direction:

$$\mathbf{s}_2 = -\mathbf{g}_2 + \beta_2\mathbf{s}_1$$

where

$$\mathbf{g}_2 = \nabla q(\mathbf{x}_2) = \mathbf{A}\mathbf{x}_2 - \mathbf{b} = \begin{pmatrix} -3/7 \\ 3/7 \end{pmatrix}$$

and

$$\beta_2 = \frac{\mathbf{g}_2^T \mathbf{g}_2}{\mathbf{g}_1^T \mathbf{g}_1} = \frac{9}{49}$$

So,

$$\mathbf{s}_2 = \begin{pmatrix} 30/49 \\ -12/49 \end{pmatrix}$$

Step size:

$$\lambda_2 = \frac{-\mathbf{g}_2^T \mathbf{s}_2}{\mathbf{s}_2^T \mathbf{A} \mathbf{s}_2} = \frac{7}{6}$$

Next point:

$$\mathbf{x}_3 = \mathbf{x}_2 + \lambda_2 \mathbf{s}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\mathbf{s}_1$ and $\mathbf{s}_2$ are conjugate, *i.e.*, $\mathbf{s}_1^T \mathbf{A} \mathbf{s}_2 = 0$, and $\mathbf{x}_3$ is the optimal solution since $\nabla q(\mathbf{x}_3) = \mathbf{A}\mathbf{x}_3 - \mathbf{b} = \mathbf{0}$.

In theory, the method has the property that, if exact arithmetic is used, convergence will occur in at most n iterations.